(total 10 points)
Due: October 17 11:59 PM.

These questions are designed for refreshing math you learned in calculus and understanding the topics discussed in class. They are divided into smaller steps for you to walk through. Each step needs no more than 5 lines of proof, so don't get too complicated.

Question 1. (4 points) This exercise shows that the 1/f-power law observed in natural images can be explained by a simple Markov Random Field MRF model.

Let $\mathbf{I}$ be an image in a continuous 2D plane (It is neat to prove this in continuous form, discrete lattice will be messy), its Fourier transform is,

$$F(\mathbf{I}) = \hat{I}(\xi, \eta) = \int\int \mathbf{I}(x, y)e^{-i2\pi(x\xi+y\eta)}dxdy.$$

$A^2(\xi, \eta) = |\hat{I}(\xi, \eta)|^2$ is the "power" of the signal at frequency component (ξ, η). We consider a MRF model with a quadratic potential $H(\mathbf{I})$.

$$p(\mathbf{I}) = \frac{1}{Z}e^{-H(\mathbf{I})}, \quad H(\mathbf{I}) = \beta\int\int(\nabla_x\mathbf{I}(x, y))^2 + (\nabla_y\mathbf{I}(x, y))^2dxdy.$$

where $\nabla_x\mathbf{I}(x, y) = \frac{\partial\mathbf{I}(x,y)}{\partial x}$ and $\nabla_y\mathbf{I}(x, y) = \frac{\partial\mathbf{I}(x,y)}{\partial y}$ are the gradient images. For boundary condition, the image $\mathbf{I}(x, y)$ is assumed to have zero intensity at infinity or to be defined on a torus.

1. Show the Fourier transform of two gradient images $\nabla_x\mathbf{I}(x, y)$ and $\nabla_x\mathbf{I}(x, y)$ are $2\pi i\xi\hat{I}$ and $2\pi i\eta\hat{I}$ respectively. That is,

$$F(\nabla_x\mathbf{I}) = 2\pi i\xi\hat{I}, \quad F(\nabla_y\mathbf{I}) = 2\pi i\eta\hat{I}.$$

   (Hint: this is the so-called Integration by Part in calculus).

2. Show that for any function $g(t)$ and its Fourier transform $G(\xi)$, we have

$$\int g(t)^2dt = \int G(\xi)^*G(\xi)d\xi$$

   $G(\xi)^*$ is the conjugate of $G(\xi)$ (as it is a complex number). Intuitively, the Fourier transform does not change the norm of a vector or function. (Hint: The proof involves switching the order of integration variables.)

3. By combining the previous two steps, show

$$H(\mathbf{I}) = 4\pi^2\beta \int \int (\xi^2 + \eta^2)|\hat{I}(\xi,\eta)|^2 d\xi d\eta.$$

What is the mean and variance for each component $\hat{I}(\xi,\eta)$?

Remark: In $\mathbf{I}$, each pixel intensity $\mathbf{I}(x,y)$ is correlated with each its neighbours, now in $\hat{I}$, each component $\hat{I}(\xi,\eta)$ is independent of other component $\hat{I}(\xi',\eta')$. it is like to be diagonalized in discrete covariance matrix in lecture.

Therefore the variance of each Fourier component $\hat{I}(\xi,\eta)$ is

$$E_p[|\hat{I}(\xi,\eta)|^2] = \frac{C}{\xi^2 + \eta^2}$$

Then we see that $A(f)$ follows the 1/f law, where $f = \sqrt{\xi^2 + \eta^2}$ is the frequency.

4. Derive the constant C above, and prove that the image has constant power $A^2(f)$ at each frequency band $[f, 2f]$, as you observed in project 1. Explain in plain language why images in this ensemble have invariant expected power-spectrum over scales (frequency bands).

Remark. This problem shows that the image ensemble defined by the MRF model $p(\mathbf{I})$ above has exact 1/f power. As we will show in lecture, it is a maximum entropy probability, it observes the 1/f-law as its sufficient statistics !

Question 2 (3 points). The goal of this exercise is to show the connection between the Gibbs/MRF model and partial differential equations (PDEs) for image processing. Consider the continuous Gibbs/MRF model for a system in problem 1 again with potential function,

$$H(\mathbf{I}(x,y)) = \int \int (\nabla_x \mathbf{I}(x,y))^2 + (\nabla_y \mathbf{I}(x,y))^2 dx dy.$$

This is the so-called functional (H is a function of function $\mathbf{I}$, and $\mathbf{I}$ is a function of position (x,y)).

Suppose we minimize the potential $H(\mathbf{I})$ by gradient descent. The dynamic of the system state is an image sequence $\mathbf{I}(x,y,t)$ showing the state changes over time,

$$\frac{d\mathbf{I}(x,y,t)}{dt} = -\frac{\delta H(\mathbf{I}(x,y,t))}{\delta \mathbf{I})}, \quad \forall x, y.$$

The right side is the derivative from variational calculus (See hint below). t is the time step. This leads to a partial differential equations (PDEs) for the system dynamics.

1. By variational calculus using the Euler-Lagrange equation, show that the PDE above is the classic *heat-diffusion equation.*

$$\frac{d\mathbf{I}(x,y,t)}{dt} = \Delta\mathbf{I}(x,y), \quad \text{or} \quad \mathbf{I}_t = \mathbf{I}_{xx} + \mathbf{I}_{yy}.$$

   where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator.

2. Rewrite the energy $H(\mathbf{I})$ in discrete form: replace the integral by summation, and the gradients by difference $\nabla_x\mathbf{I}(x,y) = \mathbf{I}(x+1,y) - \mathbf{I}(x,y)$. The derive the discrete diffusion equation for updating $\mathbf{I}(x,y,t)$ using the conventional gradient descent equation.

$$\frac{d\mathbf{I}(x,y,t)}{dt} = -\frac{dH(\mathbf{I}(x,y,t))}{d\mathbf{I}}.$$

   This actually should be a discrete form of the heat diffusion equation.

3. Suppose we use periodical boundary condition (torus), what is the image $\mathbf{I}(x,y,t)$ as $t \to \infty$.

   Remark: if we re-express the function H in the Fourier form as we did in Qestion 1.3. You can see the other way for minimizing H in the Fourier domain. Actually, Fourier transform was first invented by Joseph Fourier in 1822 to solve the heat diffusion equations. Read some background in wikipedia. In future lectures, we will see how we learn the potentials in general form and then derive system dynamics in general.

   Variational Calculus: Suppose we are minimizing a functional with respect to a function $f(x)$

$$E(f(x)] = \int L(f(x), \dot{f}(x))dx$$

   The Euler-Lagrange equation for the minimum is

$$\frac{\delta E}{\delta f} = \frac{\partial L}{\partial(f)} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{f}}\right) = 0$$

   You may find some on-line tutorial on the Euler-Lagrange equation, especially for $f(x)$ that has multiple variables $x = (x_1, ..., x_n)$. In the above question, we treat the image $\mathbf{I}(x,y)$ as a continuous function and $H(\mathbf{I})$ is a functional. In computer vision, the variational methods (PDEs) often switch to a continuous domain to derive the equations and then switch back to discrete lattice for implementation.

Question 3 (A scale invariant world, 3 points). Consider a toy world which consists of only line segments. In an image, a line segment is represented by its center (x_i, y_i), orientation θ_i and length r_i. The line segments are independently distributed with uniform

probability for their centers and orientations. The length follows a probability $p(r)$. We denote by $\lambda(a, b, A)$ the number of line segments with length $r \in [a, b]$ whose center falls inside an area A. Note that we assume the image is defined on a continuous 2D domain and the line segment has zero width (when you down scale the image, the width of the line does not change).

When we scale the image by a factor s, then the line segment will be scaled from length r to sr. Suppose that the image ensemble is scale invariant, that is, at any scale s, within a unit area, we always observe the same number (on average) of line segments with length r.

1. Show that $\lambda(a, b, A) = 4\lambda(2a, 2b, A)$. [Hint: by direct argument].

2. Use the above equation, then show that for any interval $[a, b]$,

$$\int_a^b p(r)dr = s^2 \int_{sa}^{sb} p(r)dr.$$

3. Set $a = a_o$ a constant, and $b = r$ a variable, from the above equation show $p(r) = s^3 p(sr)$, then $p(r) = \frac{c}{r^3}$.

Remark: This question proves that we will see a scale invariant world if the length of the line segments follows a distribution C/r^3!